

# Black-Scholes model under subordination

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## Abstract

In this paper we consider a new mathematical extension of the Black-Scholes model in which the stochastic time and stock share price evolution is described by two independent random processes. The parent process is Brownian, and the directing process is inverse to the totally skewed, strictly  $\alpha$ -stable process. The subordinated process represents the Brownian motion indexed by an independent, continuous and increasing process. This allows us to introduce the long-term memory effects in the classical Black-Scholes model.

## *Key words:*

Continuous-time random walk, Brownian motion, Lévy process, Subordination, Fractional calculus, Econophysics

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The option trading has the long history. The mission of the options as financial instruments is to protect investors from the stock market randomness. Since the early seventies the option market rapidly became very successive in development. The theoretical study of options was directed in finding a fair and presumably riskless price of these instruments. Without questions, the works of Black and Scholes [1] and Merton [2] are a turning-point in the study. Their method has been proven to be very useful for investors trading in option markets. On the other hand, the approach is fruitful for extending the option pricing theory in many ways. Therefore, nowadays the Black-Scholes (BS) model is very popular in finance.

The BS equation is nothing else as a diffusion equation. In fact, their option price formula is a solution of the diffusion equation with the initial and boundary conditions given by the option contract terms. The fundamental principles governing the financial and economical systems are not completely uncovered. In recent years the physical community has started applying concepts and methods of statistical and quantum physics of complex systems to

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analyze economical problems [3,4,5,6] (and references therein). The improvement of the BS model itself did not stand still too. As shown in [7], the BS equation can be derived using the Stratonovich calculus. The Gaussian assumption of the classical BS model based on ideal market conditions simplifies analytical calculations, but the empirical studies [8] show the effect of non-ideal market conditions on the true option price. In particular, the probability distribution of returns has heavy tails in contrast to a Gaussian. This explains the great interest to various generalizations of classical results. So, in [9] the stochastic dynamics of the stock and currency markets is described by the fractional Langevin-type stochastic differential equation that differs from the standard Langevin equation. The continuous-time random walk (CTRW) model is argued to provide a phenomenological description of tick-by-tick dynamics in financial markets [10]. The present paper gives arguments that the CTRW model permits ones to generalize the classical BS model. This natural extension is based on the general probabilistic formalism of limit theorems. The important preference of the approach is its analytical results. We include the long-term memory effects in the stochastic process of the BS model. The memory effects are characterized only by one parameter. To change it, one can control the contribution of memory effects to the model. The classical BS model is a particular case of the new model under the complete absence of memory.

The CTRW model is represented by two Markov processes. One of them corresponds to the random waiting-times between successive jumps, another defines the random space steps. The geometric Brownian motion is a special case of the CTRW, where time is deterministic (see below). Let  $T_1, T_2, \dots$  be non-negative and independent identically distributed (i.i.d.) random variables describing the waiting times between jumps of a walking particle. Assume that  $T_i$  belongs to the strict domain of attraction of some stable law with index  $0 < \alpha < 1$ . This means that there exist  $b_n > 0$  ( $n \in \mathbf{N}$ ), and the sum  $b_n(T_1 + \dots + T_n)$  converges in distribution to the process having the stable distribution with index  $\alpha$ . The range  $0 < \alpha < 1$  is conditioned by the support of the time steps  $T_i$  on the non-negative semi-axis. In the discrete model the internal time  $\tau$  takes on discrete values with an interval  $\delta\tau$  such that  $n \leq [\tau/\delta\tau] < n + 1$ , where  $[x]$  denotes the integer part of  $x$ . There exists the limit passage from “discrete steps” of the CTRW to “continuous steps”. The process  $b_{[\tau/\delta\tau]} \sum_{i=1}^{[\tau/\delta\tau]} T_i$  under  $\delta\tau \rightarrow 0$  converges in distribution to a new process  $T(\tau) \stackrel{d}{=} \tau^{1/\alpha} T(1)$ , where  $\stackrel{d}{=}$  means equal in distribution, and  $T(1) \stackrel{d}{=} T_1$ . The new process is Markovian, strictly  $\alpha$ -stable, totally skewed. Since  $T(\tau) \rightarrow \infty$  in probability as  $\tau \rightarrow \infty$ , the sample paths of  $\{T(\tau)\}$  are increasing almost surely (a.s.). The process  $\{T(\tau)\}$  is self-similar with exponent  $H = 1/\alpha > 1$  [11], i. e.  $\{T(c\tau)\}_{\tau \geq 0} \stackrel{f.d.}{=} \{c^{1/\alpha} T(\tau)\}_{\tau \geq 0}$  for all  $c > 0$ , where  $\stackrel{f.d.}{=}$  denotes equality of all finite dimensional distributions. Without loss of generality, we may assume jumps in the one-dimensional space. Denote by  $R_i$

the space steps. Let  $R_1, R_2, \dots$  be i.i.d. random variables independent of  $\{T_i\}$  and have the Gaussian distribution. Using the limit passage from “discrete” to “continuous” jumps, we obtain the stochastic process  $\{R(\tau)\}_{\tau \geq 0}$  with the self-similar relation  $\{R(c\tau)\}_{\tau \geq 0} \stackrel{f.d.}{=} \{c^{1/2}R(\tau)\}_{\tau \geq 0}$  for any  $c > 0$ . It should be pointed out that both processes  $\{R(\tau)\}_{\tau \geq 0}$  and  $\{T(\tau)\}_{\tau \geq 0}$  depend on the continuous internal parameter  $\tau$  that differs from the real time  $t$ .

To build the continuous position vector of the walking particle, we need the process which represents the continuous limit of the discrete counting process  $\{N_t\}_{t \geq 0}$ . For  $t \geq 0$  the number of jumps up to time  $t$  is  $N_t = \max\{n \in \mathbf{N} \mid \sum_{i=1}^n T_i \leq t\}$ , and the vector  $\mathbf{r}_{N_t} = \sum_{i=1}^{N_t} R_i$  defines the position of the particle at time  $t$ . It turns out that the scaling limit of  $\{N_t\}_{t \geq 0}$  is the hitting process of  $\{T(x)\}_{x \geq 0}$ . The hitting time process is well defined  $S(t) = \inf\{x \mid T(x) > t\}$  and depends on the true time  $t$ . The two processes  $\{T(x)\}$  and  $\{S(t)\}$  are the inverse of each other,  $S(T(\tau)) = \tau$  a.s. Since  $\{T(x)\}_{x \geq 0}$  is strictly increasing, the process  $\{S(t)\}_{t \geq 0}$  is non-decreasing. From the self-similarity of  $\{T(x)\}$  it follows the same property for  $\{S(t)\}$ , i. e.  $\{S(ct)\}_{t \geq 0} \stackrel{f.d.}{=} \{c^\alpha S(t)\}_{t \geq 0}$  for any  $c > 0$ . While  $\{T(x)\}_{x \geq 0}$  is a Lévy process, the inverse process  $\{S(t)\}_{t \geq 0}$  is no longer a Lévy process, neither a Markov process, but it is a continuous submartingal, as shown in [12]. The random value  $S(t)$  has a Mittag-Leffler distribution with  $\langle e^{-vS(t)} \rangle = \sum_{n=0}^{\infty} (-vt^\alpha)^n / \Gamma(1+n\alpha) = E_\alpha(-vt^\alpha)$ , where  $\langle X \rangle$  denotes the expectation of a real valued random variable  $X$ , and  $\Gamma(z)$  is the Gamma function. The sample paths of  $\{N_t\}_{t \geq 0}$  and  $\{S(t)\}_{t \geq 0}$  are increasing. Then the position  $\mathbf{r}_t$  of the particle at the given real time  $t$  is defined by the subordinated process  $R(S(t))$ . Recall briefly that a subordinated process  $Y(U(t))$  is obtained by randomizing the time clock of a random process  $Y(t)$  using a new clock  $U(t)$ , where  $U(t)$  is a random process with nonnegative independent increments. The resulting process  $Y(U(t))$  is said to be subordinated to  $Y(t)$ , called the parent process, and is directed by  $U(t)$ , called the directing process. The directing process is often referred to as the randomized time or operational time [13]. In general, the subordinated process  $Y(U(t))$  can become non-Markovian, though its parent process is Markovian. The process  $R(S(t))$  is self-similar with index  $\alpha/2$  such that  $\{R(S(ct))\}_{t \geq 0} \stackrel{f.d.}{=} \{c^{\alpha/2} R(S(t))\}_{t \geq 0}$  is for all  $c > 0$ . In fact, the position vector  $\mathbf{r}_t = \mathcal{B}_{S(t)}$  represents the randomization of the internal time  $\tau$  of a Brownian motion  $\mathcal{B}_\tau$  by an independent, positive and non-decreasing process  $S(t)$ .

The probability density of the position vector  $\mathbf{r}_t$  with  $t \geq 0$  satisfies

$$p^{\mathbf{r}_t}(t, x) = \int_0^\infty p^R(\tau, x) p^S(t, \tau) d\tau, \quad (1)$$

where  $p^R(\tau, x)$  represents the probability to find the parent process  $R(\tau)$  at  $x$  on the operational time  $\tau$ , and  $p^S(t, \tau)$  is the probability to be at the op-

erational time  $\tau$  on the real time  $t$ . The Laplace transform of the probability density of the random variable  $S(t)$  with respect to  $x$  gives

$$\bar{p}^S(t, v) = \int_0^\infty e^{-vx} p^S(t, x) dx = \langle e^{-vS(t)} \rangle = E_\alpha(-vt^\alpha).$$

We need also the Laplace transform of  $p^S(t, x)$  with respect to  $t$ . The Mittag-Leffler function  $E_\alpha(-vt^\alpha)$  has the following Laplace transform  $u^{\alpha-1}/(u^\alpha + v)$  with respect to  $t$ . To invert the latter analytically, we obtain

$$\hat{p}^S(u, x) = \int_0^\infty e^{-ut} p^S(t, x) dt = u^{\alpha-1} \exp\{-u^\alpha x\}.$$

In Laplace space the probability density  $p^{\mathbf{r}^i}(t, x)$  has the most simple form  $u^{\alpha-1}\hat{p}^R(u^\alpha, x)$ , as  $\hat{p}^R(u^\alpha, x) = \int_0^\infty p^R(\tau, x) \exp\{-u^\alpha \tau\} d\tau$ . For our purpose, it is useful to find the explicit form of the probability density  $p^{\mathbf{r}^i}(t, x)$ . According to the inverse formula applied to  $\hat{p}^S(u, x)$ , we have

$$p^S(t, x) = \frac{1}{2\pi i} \int_{Br} e^{ut-xu^\alpha} u^{\alpha-1} du, \quad (2)$$

where  $Br$  denotes the Bromwich path. Make the variable transform  $ut \rightarrow u$  and denote  $w = x/t^\alpha$ . Then we deform the Bromwich path into the Hankel path  $Ha$  for which a contour begins at  $u = -\infty - ia$  ( $a > 0$ ), encircles the branch cut that lies along the negative real axis and comes to the end at  $u = -\infty + ib$  ( $b > 0$ ). Expanding function  $\exp\{-wu^\alpha\}$  in a Taylor series about  $w$  and using the Hankel representation of the reciprocal of the Gamma function

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{Ha} e^u u^{-z} du,$$

we get the following series

$$p^S(t, x) = t^{-\alpha} \sum_{k=0}^{\infty} \frac{(-x/t^\alpha)^k}{k! \Gamma(1 - \alpha - k\alpha)} = t^{-\alpha} F_\alpha(x/t^\alpha).$$

Further, we briefly consider the character of  $F_\alpha(z)$ .

The function  $F_\alpha(z)$  is an entire function in  $z$ . It has the H-function representation  $H_{11}^{10} \left( z \mid \begin{smallmatrix} 1-\alpha, \alpha \\ (0,1) \end{smallmatrix} \right)$  [14]. The important property of  $F_\alpha(z)$  is that it is

non-negative for  $z > 0$ . It is easily verified that  $\int_0^\infty F_\alpha(z) dz = 1$ . Thus, the function can be a probability density. The case  $\alpha = 1$  corresponds to the Dirac  $\delta$ -function,  $F_1(z) = \delta(z - 1)$ . In particular cases  $\alpha = 1/2$  and  $\alpha = 1/3$  we have  $F_{1/2}(z) = \exp\{-z^2/4\}/\sqrt{\pi}$  and  $F_{1/3}(z) = 3^{2/3}\text{Ai}(z/3^{1/3})$  respectively, where  $\text{Ai}$  denotes the Airy function [15]. The function  $F_\alpha(z)$  has also other interesting properties. For  $0 < \alpha \leq 1/2$  the function is monotonic decreasing, whereas for  $1/2 < \alpha < 1$  it has a maximum value at a certain point  $z_{\max}$  depending on  $\alpha$ . It should be observed here that the basic Cauchy and Signaling problems of the time fractional diffusion-wave equation can be expressed in terms of the function  $F_\alpha(z)$  [16,17].

Turning back to Eq. (1), the probability density  $p^{\mathbf{r}_t}(t, x)$  is written as

$$p^{\mathbf{r}_t}(t, x) = \int_0^\infty F_\alpha(z) p^R(t^\alpha z, x) dz = \frac{1}{\sqrt{\pi D t^\alpha}} \int_0^\infty F_\alpha(z) e^{-x^2/(D t^\alpha z)} \frac{dz}{\sqrt{z}}, \quad (3)$$

where  $D$  is the constant. This function is non-negative and satisfies the normalization condition

$$\int_{-\infty}^\infty p^{\mathbf{r}_t}(t, x) dx = \int_0^\infty F_\alpha(z) dz = 1.$$

Since the parent process  $\{R(\tau)\}$  and the directing process  $\{S(t)\}$  have finite moments of any order, the subordinated process  $\{R(S(t))\}$  has finite moments of any order too. The first and second moments of  $\mathbf{r}_t$  can be obtained by the direct calculations:

$$\begin{aligned} \langle \mathbf{r}_t \rangle &= 0, \\ \langle \mathbf{r}_t^2 \rangle &= \frac{1}{2} D t^\alpha \int_0^\infty z F_\alpha(z) dz = \frac{D t^\alpha}{2\Gamma(1 + \alpha)}. \end{aligned}$$

The process  $\mathbf{r}_t$  behaves as subdiffusion ( $0 < \alpha < 1$ ). Note that the boundary case  $\alpha = 1$  may be also included in the consideration because of  $T(\tau) = \tau$  a.s. Then the hitting time process is deterministic,  $S(t) = t$ . The probability density  $p^S(\tau, t)$  degenerates in the Dirac  $\delta$ -function so that  $p^{\mathbf{r}_t}(t, x)$  becomes equal to  $p^R(t, x)$ . The constant  $D$  is interpreted as a generalized diffusion coefficient with dimension  $[D] = \text{length}^2 / \text{time}^\alpha$ .

The ordinary Brownian motion satisfies the stochastic differential equation (SDE)

$$dR(\tau) = f(R(\tau)) d\tau + g(R(\tau)) d\mathcal{B}_\tau,$$

where  $f$  and  $g$  are some functions. The process subordinated to the Brownian motion  $\{R(S(t))\}_{t \geq 0} = \mathcal{B}_{S(t)}$  is a continuous martingal and the directing process  $\{S(t)\}$  is a continuous submartingal with respect to an appropriate filtration [18]. Therefore, the subordinated process obeys the following SDE

$$d\mathbf{r}_t = f(\mathbf{r}_t) dS(t) + g(\mathbf{r}_t) d\mathcal{B}_{S(t)}.$$

In the classical BS model the evolution of the option price is governed by the Brownian motion. The well-known BS formula is of the form

$$\mathcal{C}(\tau, x) = x\Phi(d_+) - Ke^{-\beta\tau}\Phi(d_-), \quad \beta = 2r/\sigma^2,$$

where  $x$  is the share price,  $K$  the striking price,  $r$  the interest rate,  $\sigma$  the volatility, and the probability integral

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z \exp\{-y^2/2\} dy$$

is calculated for

$$d_{\pm} = (2\tau)^{-1/2} [\ln(x/K) + \tau(\beta \pm 1)].$$

If the price evolution is consequent of the subordinated process  $\mathcal{B}_{S(t)}$ , the BS formula transforms into

$$\mathcal{S}(t, x) = t^{-\alpha} \int_0^{\infty} F_{\alpha}(z/t^{\alpha}) \mathcal{C}(z, x) dz. \quad (4)$$

At  $\alpha = 1$  we obtain the classical BS formula. All financial derivatives (options of any kind, futures, forwards, etc.) have the same boundary conditions, but different either initial or final condition [19]. The detailed comparison of the various cases for this new model (4) will carry out elsewhere. The fractional extension of the BS model has been considered also in [20], but on the macroscopic basis without any microscopic dynamics presented above.

Finally, we note that the index  $\alpha$  characterizes memory effects in the subordinated process  $\mathbf{r}_t$ . Let  $L(x)$  be a time-independent Fokker-Plank operator, whose exact form is not important for the following. If the ordinary Fokker-Plank equation (FPE)  $\partial p^R(\tau, x)/\partial \tau = [L(x)p^R](\tau, x)$  describes the evolution of a Brownian particle, the probability density  $p^{\mathbf{r}_t}(t, x)$  satisfies the fractional FPE. This can be shown by simple computations. Using the relation  $\hat{p}^{\mathbf{r}_t}(u, x) = u^{\alpha-1} \hat{p}^R(u^{\alpha}, x)$  in Laplace space and acting the operator  $L(x)$

on  $\hat{p}^{\mathbf{r}^t}(u, x)$ , the Laplace image  $[L(x) \hat{p}^{\mathbf{r}^t}](u, x)$  takes the form  $u^\alpha \hat{p}^{\mathbf{r}^t}(u, x) - f(x) u^{\alpha-1}$ , where  $f(x)$  is the initial condition. The inverse Laplace transform of the latter expression gives the above-mentioned fractional FPE

$$p^{\mathbf{r}^t}(t, x) = f(x) + \frac{1}{\Gamma(\alpha)} \int_0^t d\tau (t - \tau)^{\alpha-1} [L(x) p^{\mathbf{r}^t}](\tau, x).$$

The kernel of this integral equation is a power function. It just causes the long-term memory effects in the process of interest. As shown in [17], due to such kind of memory effects, the complex nature of the microscopic behavior of stochastic systems can be transmitted to the macroscopic level of their dynamics.

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